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Ionospheric Transfer Function Tests: Line of Sight and Slab

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1 Introduction

Ionospheric transfer function (ITF) algorithms determine the effects of the ionosphere on an electromagnetic (EM) signal as it propagates through. In this report, analytic formulas are derived which can be used to compare results of ITF calculations from certain algorithms.

Three ITF algorithms are covered in this report. Line of sight (LOS) with no magnetic field effects (birefringence), LOS with magnetic field effects, and the Snell's law shell model.

The ITF algorithms covered in this report are all expressed in the frequency domain. In this way, they are applied as linear time invariant (LTI) filter functions [3].

Signals in this report are assumed to have only a single component (i.e. $\hat{x}, \hat{y}$ or $\hat{z}$ in a rectangular coordinate system). Multi-component signals can be treated simply by applying the specific ITF to each component separately.

The underlying theory of each ITF is not presented here.

2 LOS algorithms

The LOS algorithms are implemented as frequency domain filter functions for a signal that propagates from a source underneath the ionosphere to a sensor located above, along a path that does not deviate from a line connecting them. Refractive bending effects are only marginally included as higher order terms in a series expansion.

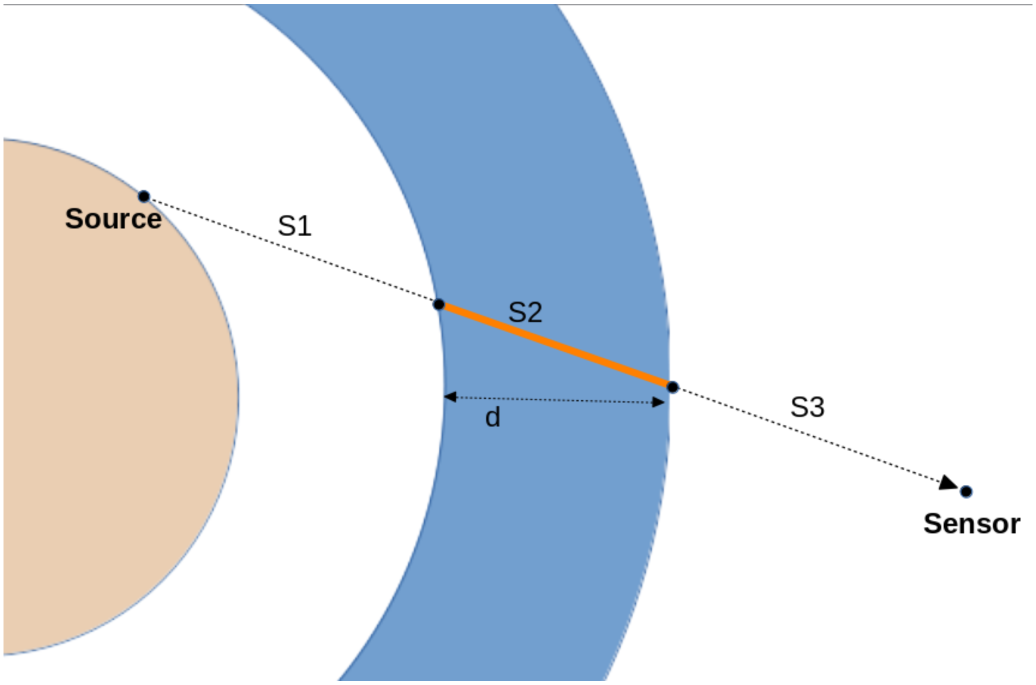

Figure 1: Simple diagram of LOS transionospheric EM signal propagation. The ionosphere is represented by the blue spherical shell region, and its effects are imposed on the signal along the orange part of its path (S_2).

Consider an EM signal originating at a source and traversing the ionosphere along a line of sight to the detector, as shown in figure 1. The ionosphere is represented by the blue region and is assumed to be a homogeneous spherical shell of some arbitrary thickness d and constant plasma electron density n_0 in a two dimensional geometry. The signal travels a total distance $S = S_1 + S_2 + S_3$, while it travels a distance $L = S_2$ through the ionosphere. Thus, the amplitude of the signal is decreased by a factor $1/S$. The Earth's magnetic field is assumed to be constant through the shell, with a magnitude of B_0. The angle β between the LOS and B_0 is therefore also a constant. Let the signal have a finite bandwidth, and assume that for each frequency component ω

$$\omega^2 > \omega_p^2 \qquad \text{and} \qquad \omega >> \omega_c \tag{1}$$

where ω_p^2 and ω_c are the plasma and electron cyclotron frequencies, respectively

$$\omega_p^2 = \frac{n_0 q^2}{\epsilon_0 m_e} \qquad \text{and} \qquad \omega_c = \frac{qB_0}{m_e} \tag{2}$$

and ϵ_0, m_e, and q are the permittivity and free space, electron mass, and electron charge in MKS units respectively.

To summarize, the assumptions used above are

- PLASMA The ionosphere is a shell of constant density plasma.
- MAGNETIC FIELD The Earth's magnetic field is a constant.
- B_0 ORIENTATION The angle between the Earth's magnetic field and the line of sight does not change.

- FREQUENCY $\omega^2 > \omega_p^2, \quad \omega >> \omega_c$.

Under the assumptions of equation 1, the $X = \omega_p^2/\omega^2$ and $Y = \omega_c/\omega$ terms in the plasma index of refraction [1] are much less than unity, and it can be Taylor expanded about $X, Y << 1$ to get

$$\mathfrak{n}(\omega) = \frac{ck(\omega)}{\omega} \approx 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2} + \frac{\mathtt{m}}{2}\frac{\omega_p^2\omega_c}{\omega^3}\cos\beta + \cdots \tag{3}$$

This can be written explicitly in terms of the wave number $k(\omega)$

$$k(\omega) \approx \frac{\omega}{c} - \frac{1}{2c}\frac{\omega_p^2}{\omega} + \frac{\mathtt{m}}{2c}\frac{\omega_p^2\omega_c}{\omega^2}\cos\beta + \cdots \tag{4}$$

where c is the speed of light and $\mathtt{m} = \pm 1$.

The time for each frequency component of the signal to traverse the distance $L = S_2$ through the ionosphere is [2]

$$t_L(\omega) = \frac{L}{v_g(\omega)} \tag{5}$$

where $v_g(\omega)$ is the frequency dependent group velocity

$$v_g(\omega) = \frac{\partial\omega}{\partial k} \tag{6}$$

so that

$$t_L(\omega) = L\frac{\partial k}{\partial\omega} \tag{7}$$

Using equation 4 this becomes

$$\begin{aligned} t_L(\omega) \;&\approx\; \frac{L}{c} + \frac{L\omega_p^2}{2c\omega^2} - \mathtt{m}\cdot\frac{L\omega_p^2\omega_c}{c\omega^3}\cos\beta + \cdots && (8)\\ &\approx\; \frac{L}{c} + \frac{q^2Ln_0}{\epsilon_0 m_e 2c\omega^2} - \mathtt{m}\cdot\frac{q^3Ln_0}{\epsilon_0 m_e^2 c\omega^3}B_0\cos\beta + \cdots && (9)\end{aligned}$$

Under the homogeneous shell assumption $Ln_0 \equiv \text{STEC}$. Using this, and the MKS values for the constants in equation 9 to third order in frequency f results in

$$t_L(f) = t_0 \;+\; t_2(f) \;+\; \mathtt{m}\cdot t_3(f) \tag{10}$$

where

$$t_0 = \frac{L}{c} \tag{11}$$

$$t_2(f) = 1.35627\cdot 10^{-7}\cdot\frac{\text{STEC}}{f^2} \tag{12}$$

$$t_3(f) = 7.67493\cdot 10^{3}\cdot\frac{\text{STEC}}{f^3}\cdot B_0\cdot\cos\beta \tag{13}$$

Equation 10 thus represents the time delay suffered by each frequency component of the signal from the bottom to the top of the ionosphere. t_0 is the time delay for each frequency over the distance L without any dispersive effects ('light delay'), $t_2(f)$ is the frequency dependent delay suffered by the signal due to only the effects of the plasma without the Earth's magnetic field, and $s \cdot t_3(f)$ represents an additional delay due to the fast ($\mathtt{m}$ = -1) or slow ($\mathtt{m} = +1$) mode dispersive effects brought about by the Earth's magnetic field (birefringence). This is shown graphically in figure 2 for a STEC of 50 TECU ($50 \cdot 10^{16}/m^2$), magnetic field magnitude of 0.25 Gauss, and angle between the line of sight and B_0 of 45^o.

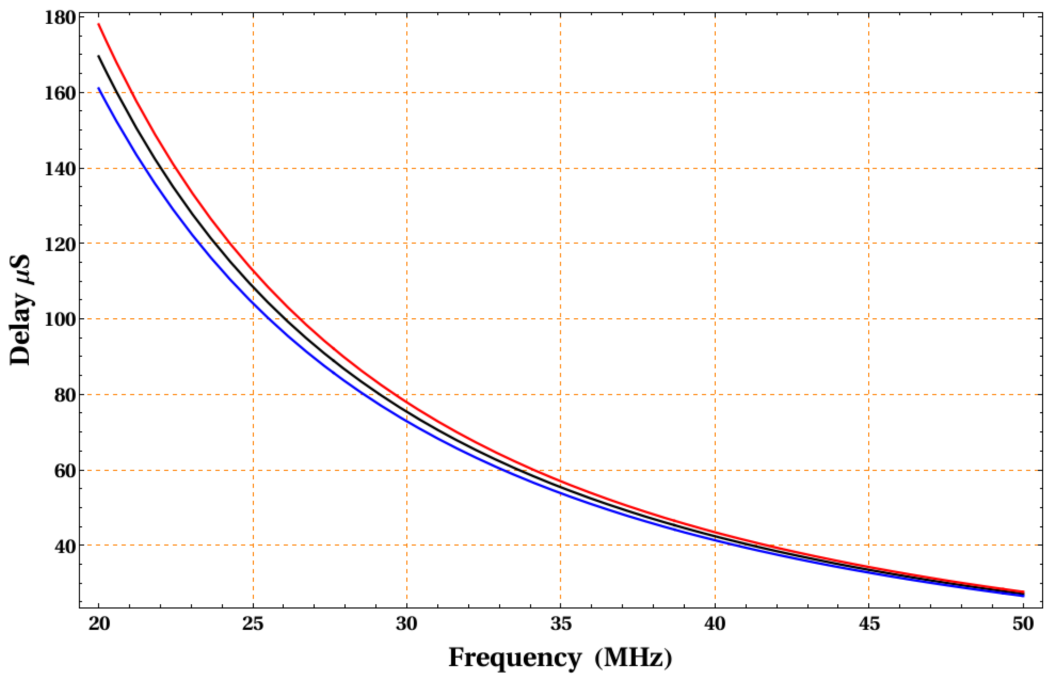

Figure 2: Time delay through the ionosphere ignoring the 'light delay'for the LOS ITF algorithms. STEC = 50 TECU, $B_0 = 0.25$ Gauss, $\beta = 45^o$. Black: $t_2(f)$ (no magnetic field effects), Red: $t_2(f)+t_3(f)$ (slow mode birefringence due to Earth's magnetic field), Blue $t_2(f) - t_3(f)$ (fast mode birefringence due to Earth's magnetic field).

3 LOS algorithms tests

For the LOS algorithms, the only parameters that determine the ionospheric effect on the signal are the STEC (determined from the plasma density n_0 and distance S_2 in figure 1, magnetic field amplitude B_0, angle β, ionospheric width d in figure 1, and distance to the sensor ($S_1 + S_2 + S_3$ in figure 1).

The amplitude of the signal is reduced by the inverse of the total path length ($S = S_1+S_2+S_3$). This amplitude reduction is different when the Earth's magnetic field is included. In that case, the each frequency component of the signal splits in to two distinct modes of propagation (fast and slow) where one, or both, could be cut off. The best way to test the amplitude calculation is to propagate only one mode ($\mathtt{m} = +1$ or $\mathtt{m} = -1$ in equation 10). In either case, the result should scale as $1/S$.

The LOS algorithms basically act on the frequency domain of the original signal as frequency dependent phase shifting filters. The phase response of a signal due to the ionosphere for these algorithms is best tested by sending a wide band signal though the ionosphere and observing a curve of maximum amplitude versus time on a frequency spectrogram of the resulting dispersed signal. The maximum of each frequency component should occur at a time corresponding to that given in equation 10, as illustrated in figure 2, where s is set to zero if the Earth's magnetic field effects are not desired. Alternatively, single frequency 'pulses' could be used to compare the delay through the ionosphere calculated from the algorithms and compared to equation 10. This has the advantage of direct comparison in the time domain without the necessity of calculating a time-frequency spectrogram.

4 Snell's law shell algorithm

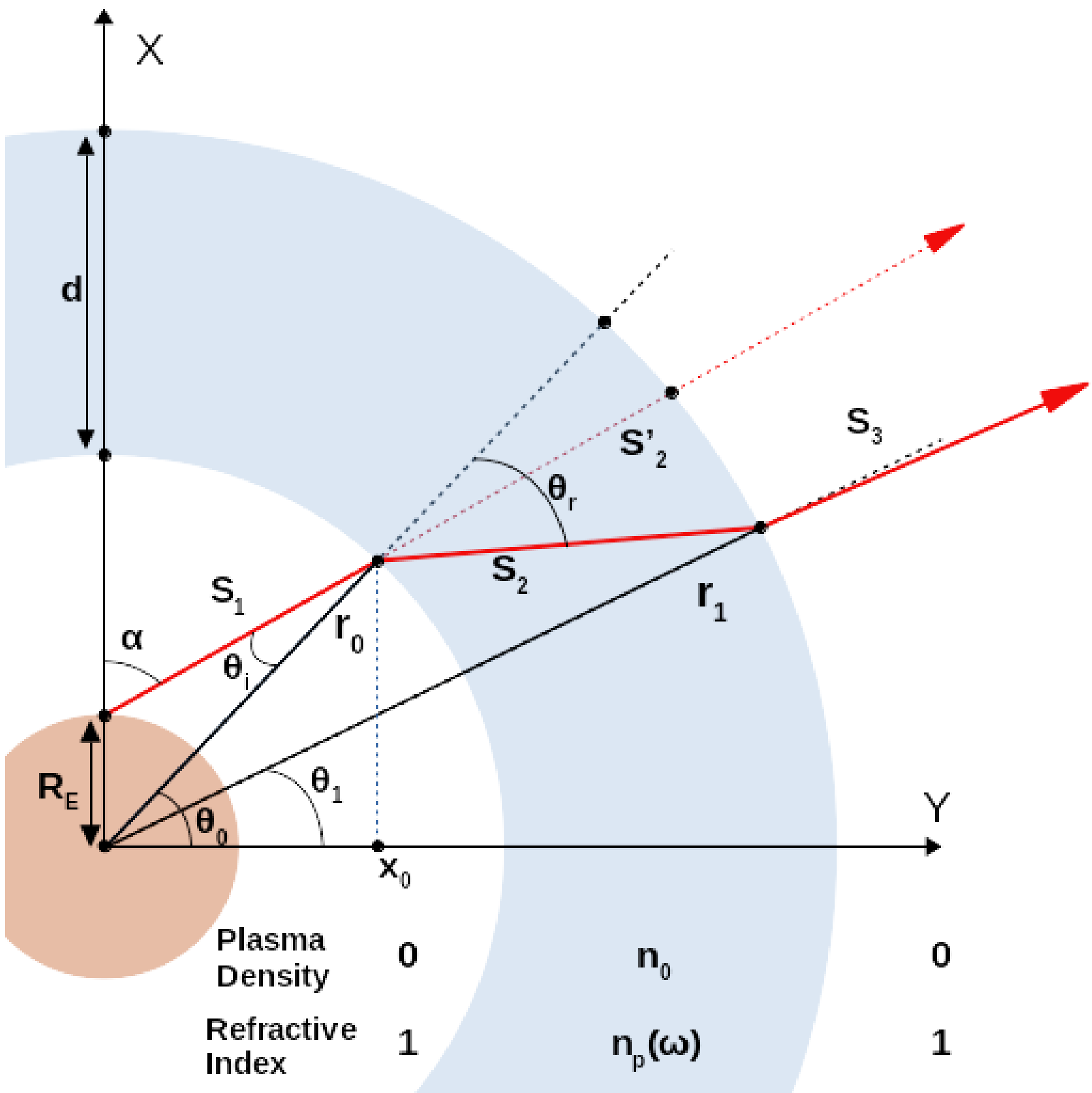

Figure 3: Geometry for the Snell's law shell model.

For this representation, the ionosphere is again assumed to be a spherical shell of constant plasma density n_0. The same assumptions from section 2 apply, except that the frequency assumption is removed so that a Taylor expansion of the index of refraction $\mathfrak{n}(\omega)$ in the plasma is no longer valid. Thus [1]

$$\mathfrak{n}(\omega) = \sqrt{1 - \frac{X(\omega)}{1 - \frac{1}{2}\frac{Y^2(\omega)\sin^2\beta}{1 - X(\omega)} + \mathtt{m}\left[\frac{1}{4}\frac{Y^4\sin^4\beta}{(1 - X(\omega))^2} + Y^2(\omega)\cos^2\beta\right]^{1/2}}} \tag{14}$$

instead of equation 3.

In addition, the two dimensional geometry is constructed so that the source is located at point P at $(0, R_E)$ where R_E is the Earth's radius. The ionosphere begins at an altitude $r_0 - R_E$ with a width d.

The difference between the LOS algorithms and this one is that the signal path (Poynting vector) is refracted according to Snell's law at the bottom and top sides of the ionosphere. The direction of propagation of the Poynting vector can be treated like a ray path under the geometric optics assumption [4, 1], which is valid for this construction and frequency range. This basically means that S_2 in figure 3 is a different length than in figure 2 for the same source location (see S_2' in figure 2).

Consider an EM signal launched at an inclination angle α at point P as shown in figure 3. Notice that the ray would travel along path S_2' using the LOS algorithm, and along path S_2 using the Snell's law shell algorithm. In what follows, the difference in STEC will be calculated for the path S_2' using the LOS algorithms and the path S_2 using the Snell's law shell algorithm

$$\Delta STEC = n_0 \cdot (S_2 - S_2') \tag{15}$$

Referring to figure 3, geometry and Snell's law are employed to reach an expression for the difference in TEC. These steps are listed below.

1. GEOMETRY

$$\frac{R_e}{\sin\theta_i} = \frac{r_0}{\sin(\pi - \alpha)} \implies \theta_i = \sin^{-1}\left(\frac{R_E}{r_0}\sin\alpha\right) \tag{16}$$

2. GEOMETRY

$$\theta_0 = \cos^{-1}\left(\frac{x_0}{r_0}\right) \tag{17}$$

3. SNELL'S LAW

$$1 \cdot \sin\theta_i = \mathfrak{n}(\omega) \cdot \sin\theta_r \implies \theta_r = \sin^{-1}\left(\frac{1}{\mathfrak{n}(\omega)}\sin\theta_i\right) \tag{18}$$

(where $n_p(\omega)$ in figure 3 has been substituted with $\mathfrak{n}(\omega)$ for clarity).

4. GEOMETRY

$$\frac{r_1}{\sin(\pi - \theta_r)} = \frac{r_0}{\sin\theta_r'} \implies \theta_r' = \sin^{-1}\left(\frac{r_0}{r_1\mathfrak{n}(\omega)}\sin\theta_i\right) \tag{19}$$

5. GEOMETRY

$$\theta_r' + (\pi - \theta_r) + \Delta\theta = \pi \implies \Delta\theta = \theta_r = \theta_r' \tag{20}$$

but $\Delta\theta = \theta_0 - \theta_1$ as well, thus

$$\theta_1 = \theta_0 - \theta_r + \theta_r' \tag{21}$$

$$\implies \theta_1 = \cos^{-1}\left(\frac{x_0}{r_0}\right) - \sin^{-1}\left(\frac{1}{\mathfrak{n}(\omega)}\sin\theta_i\right) + \sin^{-1}\left(\frac{r_0}{r_1\mathfrak{n}(\omega)}\sin\theta_i\right) \tag{22}$$

$$S_2^2 = r_0^2 + r_1^2 - 2r_0 r_1 \cos(\theta_0 - \theta_1) \tag{23}$$

from 5: $\theta_0 - \theta_1 = \theta_r - \theta_r'$

$$\implies S_2 = \sqrt{r_0^2 + r_1^2 - 2r_0 r_1 \cos\left\{\sin^{-1}\left(\frac{1}{\mathfrak{n}(\omega)}\sin\theta_i\right) - \sin^{-1}\left(\frac{r_0}{r_1\mathfrak{n}(\omega)}\sin\theta_i\right)\right\}} \tag{24}$$

Using equation 15, results in

$$\Delta STEC = n_0 \cdot \left(\sqrt{r_0^2 + r_1^2 - 2r_0 r_1 \cos\left\{\sin^{-1}\left(\frac{1}{\mathfrak{n}(\omega)}\sin\theta_i\right) - \sin^{-1}\left(\frac{r_0}{r_1\mathfrak{n}(\omega)}\sin\theta_i\right)\right\}} - S_2'\right) \tag{25}$$

Note that $\mathfrak{n}(\omega)$ depends not only on frequency, but also the factor $\mathtt{m}$, shown explicitly in equation 14. Thus, this algorithm can be implemented with and without the Earth's magnetic field.

5 Snell's law shell algorithm tests

For this algorithm, the parameters that determine the ionospheric effect on the signal are the STEC determined from the plasma density n_0 and distance S_2 in figure 3, magnetic field amplitude B_0, angle β, ionospheric width d, distance to the sensor ($S = S_1 + S_2 + S_3$ in figure 3), and the launch angle α relative to the vertical axis. Exercising this particular algorithm for the purpose of testing need only be a function of the launch angle α. There is no need to solve for the ray that gets to a predetermined sensor location. Simply choose a distance S_3 at which to calculate the results.

Again, the amplitude of the signal is reduced by the inverse of the total path length ($S = S_1 + S_2 + S_3$). This amplitude reduction is different when the Earth's magnetic field is included. In that case, each frequency component of the signal splits into two distinct modes of propagation (fast and slow) where one, or both, could be cut off. The best way to test the amplitude calculation is to propagate only one mode ($\mathtt{m} = +1$ or $\mathtt{m} = -1$ in equation 10). In either case, the result should scale as $1/S$.

This algorithm also acts on the frequency domain of the original signal as a frequency dependent phase shifting filter. However, there is not a exact analytic expression like equation 10 to determine the time delay for each frequency component. Rather, equation 25 can be used as a test of the phase response of the signal. For a given mode $\mathtt{m}$, the distance S_2 can be calculated from equation 24 (ignoring the nonphysical root). The distance S_2' that the ray would have traveled along its line of sight, as shown in figure 4, is calculated as follows. from the law of cosines

$$(r_0 + d)^2 = r_0^2 + S_2'^2 - 2r_0 S_2' \cos\epsilon \tag{26}$$

where

$$\epsilon = \pi - \theta_i \tag{27}$$

also

$$\frac{\sin\theta_i}{R_e} = \frac{\sin(\pi - \alpha)}{r_0} \qquad \implies \qquad \theta_i = \sin^{-1}\left(\frac{R_E}{r_0}\sin\alpha\right) \tag{28}$$

so that S_2' can be calculated from α since, from equation 27,

$$\epsilon = \pi - \sin^{-1}\left(\frac{R_E}{r_0}\sin\alpha\right) \tag{29}$$

and thus

$$S_2' = r_0 \cos\epsilon \pm \left[r_0^2 \cos^2\epsilon + d(2r_0 - d)\right]^{1/2} \tag{30}$$

where the nonphysical root is ignored.

Finally, S_2 (equation 24) and S_2' (equation 30) are used in equation 15 to solve for the difference in STEC.

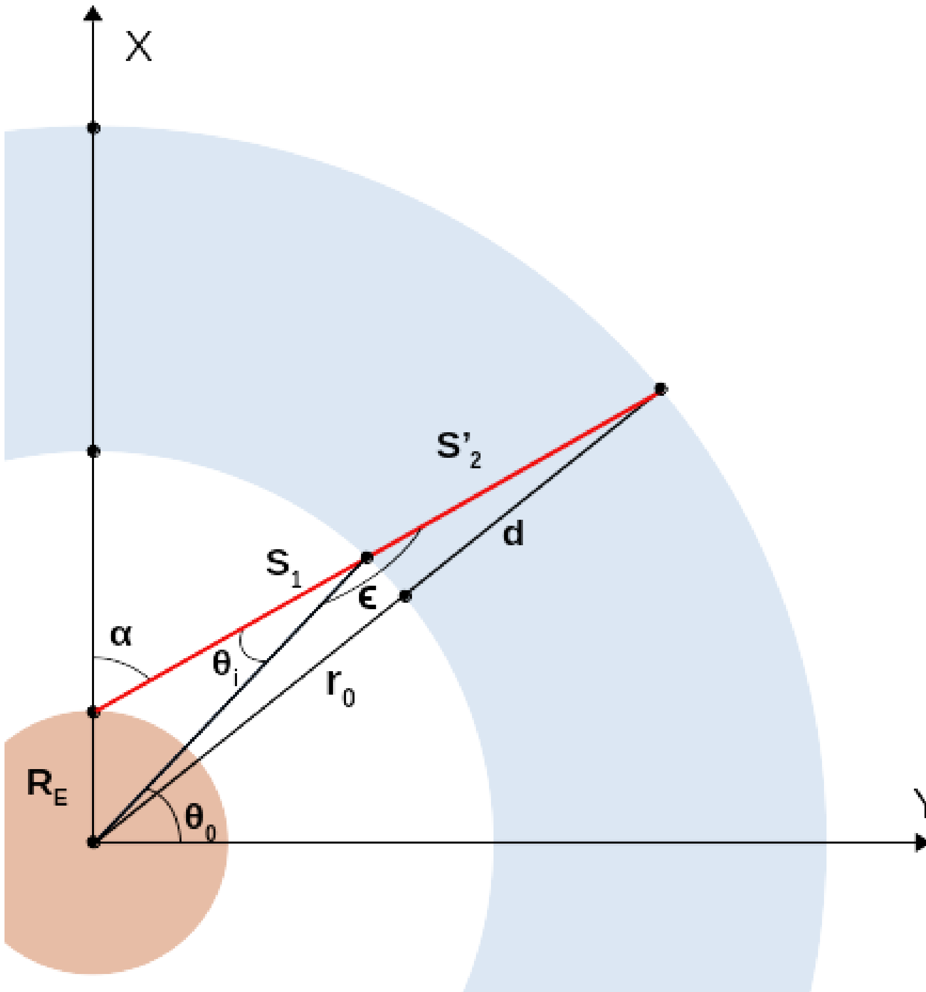

Figure 4: Geometry for calculating S_2' in the Snell's law shell model.

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